Uncertainty and Probability in rock slope stability assessment and design

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A thorough investigation of slope design criteria in open pit mining led Wesseloo and Read (in: Read and Stacey, 2009) to develop the following summary:

<table>
<thead>
<tr>
<th>Slope scale</th>
<th>Failure consequence</th>
<th>FoS (min) (static)</th>
<th>FoS (min) (Dynamic)</th>
<th>PoF (max) [FoS&lt;1]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bench</td>
<td>Low-high</td>
<td>1.1</td>
<td>NA</td>
<td>20 – 50%</td>
</tr>
<tr>
<td>Inter-ramp</td>
<td>Low</td>
<td>1.15 - 1.2</td>
<td>1.0</td>
<td>25%</td>
</tr>
<tr>
<td></td>
<td>Medium</td>
<td>1.2</td>
<td>1.0</td>
<td>20%</td>
</tr>
<tr>
<td></td>
<td>High</td>
<td>1.2 – 1.3</td>
<td>1.1</td>
<td>10%</td>
</tr>
<tr>
<td>Overall</td>
<td>Low</td>
<td>1.2 – 1.3</td>
<td>1.0</td>
<td>15 – 20%</td>
</tr>
<tr>
<td></td>
<td>Medium</td>
<td>1.3</td>
<td>1.05</td>
<td>5 – 10%</td>
</tr>
<tr>
<td></td>
<td>High</td>
<td>1.3 – 1.5</td>
<td>1.1</td>
<td>&lt; 5%</td>
</tr>
</tbody>
</table>

- It is intuitive that FoS increase (PoF decrease) for larger slopes
- But we also know: as long as strength is higher than stresses -> OK
- 1) How is this consistent?  2) What is the role of performance?
Two important considerations to answer the first question:

- Our “confidence” on our assessment of the slope -> Failure likelihood
- How much it will “hurt” given the slope fails -> Failure consequence

Therefore the criteria, as we understand it now, is a form of risk assessment.

Performance becomes a means of validating/calibrating our understanding of the slope behavior and therefore allows us to become more “confident” of our assessment of the slope. This gives room for increasing the efficiency of slope design (lower FoS or constant PoF for steeper slopes).

Key points:

- The level of confidence on our slope assessments is driven by the uncertainty in our analyses.
- There are different approaches to manage uncertainty, which will reflect on the criteria selected for slope evaluation.
Today:

- Sources of uncertainty in rock slope engineering
  - Model uncertainty
  - Parameter uncertainty
  - Human uncertainty
- Uncertainty and probability in slope assessments
  - Notions of probability
  - Quantifying parameter uncertainty
  - Propagating uncertainty in slope analyses
  - Reliability, FoS and PoF
- Role of monitoring and performance
- Conclusions
Today:

Keep in mind the life of the pit

Read and Stacey, 2009
Sources of uncertainty in rock slope engineering

Sources of uncertainty

• There are a number of classifications of uncertainty (or classification of sources of uncertainty) in the literature.

• One classification that proves adequate for the geotechnical practitioner is presented below:

• This classification becomes clear for the geotechnical practitioner as it applies to the analysis of slopes. It addresses the potential for slope failure.

• What about the consequences?
Sources of uncertainty in rock slope engineering

**Model uncertainty**

- We can discuss two types of models
  - 1) Our mental models of reality (theory)
  - 2) Models for analysis (calculation tools)

- The questions associated with these sources of uncertainty include:
  - Is this a valid theory for the situation?
  - What behavior should I expect during excavation? After?
  - Are the simplifications adequate for analysis? Do we capture the expected behavior?
Sources of uncertainty in rock slope engineering

Read and Stacey, 2009
Sources of uncertainty in rock slope engineering

Model uncertainty

Geological models

Investigation data

Structural models

Martin and Stacey, 2017

Read and Stacey, 2009
Sources of uncertainty in rock slope engineering

Model uncertainty

Geological models

Hydrogeological models

Structural models

Stability models

Read and Stacey, 2009
Martin and Stacey, 2017
Sources of uncertainty in rock slope engineering

Model uncertainty

Expected behavior:

- Dilate? Contract?
- Collapsible material? Swelling potential?
- Mohr-Coulomb? Hoek-Brown? Discontinuous?
- Progressive failure?
- Ductile? Brittle?

Tools available for reducing model uncertainty:

- Peer reviews
- Case history analysis (past performance)

Expected mode of failure:

Martin and Stacey, 2017
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Parameter uncertainty

Data Scatter
- Real Spatial Variability
- Random Testing Error

Systematic Error
- Statistical Error
- Bias in Measurements

after El-Ramly, 2001
Sources of uncertainty in rock slope engineering

- Parameter uncertainty
- Spatial variability from heterogeneous weathering

- Variability with depth.
- Similarly, horizontal variability
- Both depend on the geological history!
- What about in the rock world?

Phoon and Kulhawy, 1999

Martin and Stacey, 2017
Sources of uncertainty in rock slope engineering

- Difficulties in the rock world (and some soils) from the complex morphologic history when compared with sediments (alteration, structures, etc.)

- Therefore, Our conceptual models vary with the dimension of the slopes

Read and Stacey, 2009

Martin and Stacey, 2017
Sources of uncertainty in rock slope engineering

- Multiple approaches with varying degrees of complexity based on non-linearity and multiple parameters
- Basic method is a linear regression:

\[ x_i = a_0 + a_i z_i + e_i \]

\[ a_0 = E[x] - a_i E[z] \]

\[ a_i = \frac{\sum(z_i - E[z])(x_i - E[x])}{\sum(z_i - E[z])^2} \]

\[ e_i = ? \]
Sources of uncertainty in rock slope engineering

Parameter uncertainty

- The trends are considered deterministic, while residuals (e) are treated stochastically.
- Spatial variability then focuses on correlations between residuals.

El-Ramly, 2001
Sources of uncertainty in rock slope engineering

Parameter uncertainty

- Autocovariance $C_x(r)$: measure of data $x$ “similarity” for a given distance $r$

\[
C_x(r) = E[(x_i - t_i)(x_{i+r} - t_{i+r})]
\]

- Small $r$, large $C_x(r)$. Decreases with increasing $r$. 

El-Ramly, 2001
Sources of uncertainty in rock slope engineering

Parameter uncertainty

- **Semivariograms** $\gamma_x(r)$: Commonly used in geostatistics for mining and reservoir characterization.

![Semivariogram Diagram]

- Typically, assumes stationary data average (no trend), or simple spatial trend.
- Average measure of dissimilarity between data separated by a distance ($r$).

\[
\gamma_x(r) = \frac{1}{2} E[(x_i - x_{i+r})^2]
\]

El-Ramly, 2001
Sources of uncertainty in rock slope engineering

What are these telling us?

- Trends define large scale spatial variability
- \( r \) defines our understanding of small scale variability vs. random variability
- These values are estimated based on observations (tests) through simple trend analyses (e.g. least squares), estimates of mean and variance (method of moments) for errors around the trend.
- Take advantage of CPT for soil-type behavior
- Do we have the information for these analyses ref. geotechnical parameters in rock?
- -> We work based on geotechnical domains and we assign geotechnical parameters for each.
Sources of uncertainty in rock slope engineering

Parameter uncertainty

So now we select our domains, we go get some more stuff drilled, we have fun characterizing the discontinuities, and we get some core tested... In the example below; What do we use for UCS? Apply statistical techniques?
Sources of uncertainty in rock slope engineering

Care with blind quantification of variability without due regard for the physical reality!

Spatial variability can have a significant effect on model response, even if the same statistical characteristics are used.

El-Ramly, 2001
Sources of uncertainty in rock slope engineering

Parameter uncertainty

Input of engineering judgment Required for: 1) subdividing the domain, 2) characteristic value of the parameter, 3) working range of the parameter, 4) distribution of parameter values.

Example variability in basic parameters for different rocks:

<table>
<thead>
<tr>
<th>Rock</th>
<th>$\sigma_e$ (MPa)</th>
<th>E (GPa)</th>
<th>$\nu$</th>
<th>Rock</th>
<th>$\sigma_e$ (MPa)</th>
<th>E (GPa)</th>
<th>$\nu$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Andesite</td>
<td>120–320</td>
<td>30–40</td>
<td>0.20–0.30</td>
<td>Granodiorite</td>
<td>100–200</td>
<td>30–70</td>
<td>0.15–0.30</td>
</tr>
<tr>
<td>Amphibolite</td>
<td>250–300</td>
<td>30–90</td>
<td>0.15–0.25</td>
<td>Greywacke</td>
<td>75–220</td>
<td>20–60</td>
<td>0.05–0.15</td>
</tr>
<tr>
<td>Anhydrite</td>
<td>80–130</td>
<td>50–85</td>
<td>0.20–0.35</td>
<td>Gypsum</td>
<td>10–40</td>
<td>15–35</td>
<td>0.20–0.35</td>
</tr>
<tr>
<td>Basalt</td>
<td>145–355</td>
<td>35–100</td>
<td>0.20–0.35</td>
<td>Limestone</td>
<td>50–245</td>
<td>30–65</td>
<td>0.25–0.35</td>
</tr>
<tr>
<td>Diabase</td>
<td>240–485</td>
<td>70–100</td>
<td>0.25–0.30</td>
<td>Marble</td>
<td>60–155</td>
<td>30–65</td>
<td>0.25–0.40</td>
</tr>
<tr>
<td>Diorite</td>
<td>180–245</td>
<td>25–105</td>
<td>0.25–0.35</td>
<td>Quartzite</td>
<td>200–460</td>
<td>75–90</td>
<td>0.10–0.15</td>
</tr>
<tr>
<td>Dolerite</td>
<td>200–330</td>
<td>30–85</td>
<td>0.20–0.35</td>
<td>Sandstone</td>
<td>35–215</td>
<td>10–60</td>
<td>0.10–0.45</td>
</tr>
<tr>
<td>Dolomite</td>
<td>85–90</td>
<td>44–51</td>
<td>0.10–0.35</td>
<td>Shale</td>
<td>35–170</td>
<td>5–65</td>
<td>0.20–0.30</td>
</tr>
<tr>
<td>Gabbro</td>
<td>210–280</td>
<td>30–65</td>
<td>0.10–0.20</td>
<td>Siltstone</td>
<td>35–250</td>
<td>25–70</td>
<td>0.20–0.25</td>
</tr>
<tr>
<td>Gneiss</td>
<td>160–200</td>
<td>40–60</td>
<td>0.20–0.30</td>
<td>Slate</td>
<td>100–180</td>
<td>20–80</td>
<td>0.15–0.35</td>
</tr>
<tr>
<td>Granite</td>
<td>140–230</td>
<td>30–75</td>
<td>0.10–0.25</td>
<td>Tuff</td>
<td>10–45</td>
<td>3–20</td>
<td>0.20–0.30</td>
</tr>
</tbody>
</table>

Source: Data selected from Jaeger & Cook (1979), Goodman (1989), Bell (2000), Gonzalez de Vallejo (2002)

Read and Stacey, 2009

- A measure of this variability is the Coefficient of Variation (COV).

\[ COV = \frac{s}{\mu} \]

\[ \mu_x = \frac{\sum_{i=1}^{n} x_i}{n} \quad s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n-1}} \]
**Sources of uncertainty in rock slope engineering**

*Parameter uncertainty*

**Examples of COV**

\[
COV = \frac{s}{\mu}
\]

<table>
<thead>
<tr>
<th>Test type</th>
<th>Property</th>
<th>Material type</th>
<th>Coefficient of variation (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>m</td>
</tr>
<tr>
<td>Index</td>
<td>$\gamma$, $\gamma_d$</td>
<td>fine-grained</td>
<td>14</td>
</tr>
<tr>
<td></td>
<td>$w_n$</td>
<td>fine-grained</td>
<td>40</td>
</tr>
<tr>
<td></td>
<td>$w_p$</td>
<td>fine-grained</td>
<td>23</td>
</tr>
<tr>
<td></td>
<td>$w_L$</td>
<td>fine-grained</td>
<td>38</td>
</tr>
<tr>
<td></td>
<td>PI - all data</td>
<td>fine-grained</td>
<td>33</td>
</tr>
<tr>
<td></td>
<td>- $\leq 20%$</td>
<td>fine-grained</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>- $&gt; 20%$</td>
<td>fine-grained</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>$\gamma$, $\gamma_d$</td>
<td>rock</td>
<td>42</td>
</tr>
<tr>
<td></td>
<td>n</td>
<td>rock</td>
<td>25</td>
</tr>
<tr>
<td>Strength</td>
<td>$\phi'$, tan $\phi'$</td>
<td>sand, clay</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td></td>
<td>sand</td>
<td>32</td>
</tr>
<tr>
<td></td>
<td></td>
<td>clay</td>
<td>16</td>
</tr>
<tr>
<td></td>
<td>$s_u$</td>
<td>clay</td>
<td>100</td>
</tr>
<tr>
<td></td>
<td>$q_u$</td>
<td>rock</td>
<td>178</td>
</tr>
<tr>
<td></td>
<td>$q_{lt}$-brazilian</td>
<td>rock</td>
<td>74</td>
</tr>
<tr>
<td>Stiffness</td>
<td>$E_{t-50}$</td>
<td>rock</td>
<td>32</td>
</tr>
<tr>
<td>VST</td>
<td>$s_u$(VST)</td>
<td>clay</td>
<td>26</td>
</tr>
<tr>
<td>DMT</td>
<td>$E_D$ - all data</td>
<td>sand</td>
<td>31</td>
</tr>
<tr>
<td></td>
<td>- w/o outliers</td>
<td>sand</td>
<td>30</td>
</tr>
</tbody>
</table>

Kulhawy et al. 2000
Sources of uncertainty in rock slope engineering

- The engineer needs to decide the approach to deal with parameter uncertainty: Characteristic values? Sensitivity analyses? Probabilistic approach? Observational method?

- These treated inherent variability – there is also bias and testing error:
  
  point load test $\rightarrow$ UCS $\rightarrow$ Triaxial test $\rightarrow$ field test

  Rock mass parameters through Q, GSI, RMR

- When does the largest impact of parameter uncertainty occur in the life of the open pit slopes?
Sources of uncertainty in rock slope engineering

Read and Stacey, 2009
Sources of uncertainty in rock slope engineering

- Human uncertainty
  - Most difficult to address
  - Skill set, work ethics, company culture, etc.
  - Tools: communication, peer reviews, safety culture optimization
  - Very difficult to quantify, we take a management approach through Safety Management Systems

Example Safety Management System elements (ESRM)

1. Management Leadership, Commitment and Accountability.
2. Risk Assessment and Management of Risks.
5. Incident Reporting, Investigation, Analysis and Actions.
6. Program Evaluation (Safety Audits) and Continuous Improvement.
7. Design and Construction.
9. Employee Competency and Training.
10. Contractor Competency and Integration.
Sources of uncertainty in rock slope engineering

Read and Stacey, 2009
Sources of uncertainty in rock slope engineering

The trick to manage human uncertainty is to map these two into the design-construction-operation-closure

1) Management Leadership, Commitment and Accountability.
2) Risk Assessment and Management of Risks.
4) Management of Change.
5) Incident Reporting, Investigation, Analysis and Actions.
6) Program Evaluation (Safety Audits) and Continuous Improvement.
7) Design and Construction.
8) Operations and Maintenance.
9) Employee Competency and Training.
10) Contractor Competency and Integration.
11) Operations and Facilities Information and Documentation.

Read and Stacey, 2009
Sources of uncertainty in rock slope engineering

Can we quantify the uncertainty from all sources?
Do we need to?

Summarizing some key point the previous slides for dealing with uncertainty:

- **Model uncertainty**: through peer reviews, case studies
- **Human uncertainty**: Strong Safety Management Systems
- **Parameter Uncertainty**: statistical / probabilistic

This last one provides an opportunity for performance-based approaches and implementation of the Observational Method (a form of Bayesian updating)
Uncertainty and probability in slope assessments

Now you might be asking:

• How do we quantify uncertainty such that it can be propagated to reflect the likelihood of a slope failure? -> Probability of Failure (PoF)
• What is the relationship between the PoF and the Fos?
• How can slope performance influence criteria for PoF or FoS?

Let’s briefly review some concepts of probability

*It is not the intent to provide a primer in probability theory, but to provide a basic common understanding for our discussions.*
Uncertainty and probability in slope assessments

Key notions of probability:

1) Probability is a quantitative measure of likelihood, with values between 0 (impossible) and 1 (certain).

2) Probability can measure the ratios between possible states of a system (e.g. the probability of obtaining the number 3 when tossing a die is 1/6).

3) Probability of an outcome is the number of times the outcome was observed divided by the total number of tests (frequency approach).

All apply to the evaluation of rock slopes
Uncertainty and probability in slope assessments

Let us use UCS results as an example:
No. tests: 30
Results (in MPa):

\[
\mu_x = \frac{\sum_{i=1}^{n} x_i}{n}
\]
\[
s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - \mu_x)^2}{n - 1}}
\]

\[
COV = \frac{s}{\mu}
\]

<table>
<thead>
<tr>
<th>Value</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>1</td>
</tr>
<tr>
<td>179</td>
<td>1</td>
</tr>
<tr>
<td>180</td>
<td>1</td>
</tr>
<tr>
<td>120</td>
<td>1</td>
</tr>
<tr>
<td>145</td>
<td>1</td>
</tr>
<tr>
<td>172</td>
<td>1</td>
</tr>
<tr>
<td>167</td>
<td>1</td>
</tr>
<tr>
<td>115</td>
<td>1</td>
</tr>
<tr>
<td>163</td>
<td>1</td>
</tr>
<tr>
<td>136</td>
<td>1</td>
</tr>
<tr>
<td>129</td>
<td>1</td>
</tr>
<tr>
<td>157</td>
<td>1</td>
</tr>
<tr>
<td>179</td>
<td>1</td>
</tr>
<tr>
<td>143</td>
<td>1</td>
</tr>
<tr>
<td>119</td>
<td>1</td>
</tr>
</tbody>
</table>

All data
- mean: 144.5
- St. dev.: 32.72
- COV: 0.2265

Reduced data
- mean: 143.8
- St. dev.: 20.21
- COV: 0.1406
Uncertainty and probability in slope assessments

Let us use UCS results as an example:

\[
\begin{array}{|c|c|c|}
\hline
& All data & Reduced data \\
\hline
\text{mean} & 144.5 & 143.8 \\
\text{St. dev.} & 32.72 & 20.21 \\
\text{COV} & 0.2265 & 0.1406 \\
\hline
\end{array}
\]

What questions would you ask?
Uncertainty and probability in slope assessments

Another example: GSI for Bighorn Sandstone?

[Geological Strength Index (GSI) chart and image of rock formations labeled as Upper Bighorn Sandstone and Bighorn Siltstone]
Uncertainty and probability in slope assessments

Another example:

GSI for Bighorn Sandstone, between 40 and 60?

Refresher on common distributions

Discrete:
- Binomial
- Bernoulli
- Poisson
- Discrete

Continuous:
- Uniform
- Normal
- Triangular
- Gamma
- Log-Normal
- Exponential
- Pearson
- ...
- Keep going!

Goodness of fit through visual, Q-Q plots, Chi-square tests, etc.
Uncertainty and probability in slope assessments

Some discrete distributions

<table>
<thead>
<tr>
<th>Name</th>
<th>Distribution Function $F(t) = \Pr(\tau \leq t)$</th>
<th>Density $f(t) = dF(t)/dt$</th>
<th>Parameter Range</th>
<th>Mean $E[\tau]$</th>
<th>Variance $\text{Var}[\tau]$</th>
</tr>
</thead>
</table>
| **Binomial**    | $\Pr(\zeta \leq k) = \sum_{i=0}^{k} p_i$       | $p_i = \binom{n}{i} p^i (1-p)^{n-i}$ | $k = 0, \ldots, n$  
$0 < p < 1$ | $np$ | $np(1-p)$ |
| **Poisson**     | $\Pr(\zeta \leq k) = \sum_{i=0}^{k} p_i$       | $p_i = \frac{m^i}{i!} e^{-m}$ | $k = 0, 1, \ldots$  
$m > 0$ | $m$ | $m$ |
| **Geometric**   | $\Pr(\zeta \leq k) = \sum_{i=1}^{k} p_i = 1-(1-p)^k$ | $p_i = p(1-p)^{i-1}$ | $k = 1, 2, \ldots$  
$0 < p < 1$ | $\frac{1}{p}$ | $\frac{1-p}{p^2}$ |
| **Hypergeometric** | $\Pr(\zeta \leq k) = \sum_{i=0}^{k} \frac{\binom{K}{i} \binom{N-K}{n-i}}{\binom{N}{n}}$ | | $k = 0, 1, \ldots$  
$\min(K, n)$ | $\frac{K}{N}$ | $\frac{K n (N-K) (N-n)}{N^2 (N-1)}$ |
### Uncertainty and probability in slope assessments

#### Some continuous distributions

<table>
<thead>
<tr>
<th>Name</th>
<th>Distribution Function $F(t) = \Pr[\tau \leq t]$</th>
<th>Density $f(t) = dF(t)/dt$</th>
<th>Parameter Range</th>
<th>Mean $\mathbb{E}[\tau]$</th>
<th>Variance $\text{Var}[\tau]$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exponential</td>
<td>$1 - e^{-\lambda t}$</td>
<td><img src="graph1.png" alt="Graph" /></td>
<td>$t &gt; 0$ ($F(t)=0$, $t \leq 0$) $\lambda &gt; 0$</td>
<td>$\frac{1}{\lambda}$</td>
<td>$\frac{1}{\lambda^2}$</td>
</tr>
<tr>
<td>Weibull</td>
<td>$1 - e^{- (\lambda t)^\beta}$</td>
<td><img src="graph2.png" alt="Graph" /></td>
<td>$t &gt; 0$ ($F(t)=0$, $t \leq 0$) $\lambda, \beta &gt; 0$</td>
<td>$\frac{\Gamma(1+\frac{1}{\beta})}{\lambda}$</td>
<td>$\frac{\Gamma(1+\frac{2}{\beta}) - \Gamma^2(1+\frac{1}{\beta})}{\lambda^2}$</td>
</tr>
<tr>
<td>Gamma</td>
<td>$\frac{\lambda t}{\Gamma(\beta)} \int_0^t x^{\beta-1} e^{-x} dx$</td>
<td><img src="graph3.png" alt="Graph" /></td>
<td>$t &gt; 0$ ($F(t)=0$, $t \leq 0$) $\lambda, \beta &gt; 0$</td>
<td>$\frac{\beta}{\lambda}$</td>
<td>$\frac{\beta}{\lambda^2}$</td>
</tr>
<tr>
<td>Chi-square ($\chi^2$)</td>
<td>$\frac{t^{v/2-1} e^{-t/2}}{2^{v/2} \Gamma(v/2)}$</td>
<td><img src="graph4.png" alt="Graph" /></td>
<td>$t &gt; 0$ ($F(t)=0$, $t \leq 0$) $v = 1, 2, ...$ (degrees of freedom)</td>
<td>$v$</td>
<td>$2v$</td>
</tr>
<tr>
<td>Normal</td>
<td>$\frac{1}{\sigma \sqrt{2 \pi}} \int_{-\infty}^t e^{-\frac{(x-m)^2}{2 \sigma^2}} dx$</td>
<td><img src="graph5.png" alt="Graph" /></td>
<td>$\infty &lt; t$, $m &lt; \infty$ $\sigma &gt; 0$</td>
<td>$m$</td>
<td>$\sigma^2$</td>
</tr>
<tr>
<td>Lognormal</td>
<td>$\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^t \frac{\ln(\lambda t)}{\sigma} e^{-x^2/2} dx$</td>
<td><img src="graph6.png" alt="Graph" /></td>
<td>$t &gt; 0$ ($F(t)=0$, $t \leq 0$) $\lambda, \sigma &gt; 0$</td>
<td>$\frac{e^{\sigma^2/2}}{\lambda}$</td>
<td>$\frac{e^{2\sigma^2} - e^{\sigma^2}}{\lambda^2}$</td>
</tr>
</tbody>
</table>
Uncertainty and probability in slope assessments

We have now a quantitative approach for parameter uncertainty within our geotechnical domains, but; how do we propagate this uncertainty and estimate a PoF?

- In a simple design for stability, we want the slope to be stable.
- That is, we want the resistance component \( R \) > load component \( Q \)

In slope design we know that a FoS \( \leq 1 \) represents instability:

\[
 FoS = \frac{R}{Q} \leq 1 \\
(Sometimes used 1 – FoS = 1 – \frac{R}{Q} \leq 0)
\]

What we are trying to do:

PDF of input parameters \( \rightarrow \) PDF of Loads \( \rightarrow \) Slope analysis \( \rightarrow \) PDF of FoS

PoF
Uncertainty and probability in slope assessments

How do we arrive to the distributions of R and Q, and ultimately of FoS from the parameter probability distributions?

• For simple expressions, close form solutions can provide the probability distribution of the outcome. However, even simple limit equilibrium equations are not suitable for this approach.

• Common methods for uncertainty propagation:
  • First Order Second Moment Method
  • Point Estimate Method
  • Monte Carlo-type Simulations

We will provide a brief description for reference
Uncertainty and probability in slope assessments

First Order Second Moment Method

Given that the mathematical expression for the Factor of Safety expressed as:

\[ g(x_1, x_2, \ldots x_n); \]

First order second moment (FOSM) is based on Taylor’s series expansion of a \( g(x_1, x_2, \ldots x_n) \) around its mean value. For simplicity, only the linear terms of the series are usually retained. The mean and variance of the safety factor are given by

\[ E[FS] = g(E[x_1], E[x_2], \ldots E[x_n]) \]

\[ V[FS] \approx \sum_{i=1}^{k} \sum_{j=1}^{k} \left( \frac{\partial g}{\partial x_i} \frac{\partial g}{\partial x_j} \right) C(x_i, x_j) \]

where \( E[FS] \) and \( V[FS] \) are the mean and variance of the safety factor, respectively, and \( C(x_i, x_j) \) is the covariance between input variables \( x_i \) and \( x_j \). Detailed description of FOSM is available in many text books such as Ang and Tang (1984) and Harr (1977 and 1987).
Uncertainty and probability in slope assessments

Point Estimate Method

It is based on replacing the continuous probability distribution of input variables with discrete distributions with two values and associated probabilities such that the mean and variance of the original and discrete distributions are the same.

The mean and variance of the safety factor are evaluated by adding $2^n$ estimates of the performance function, where $n$ is the number of input variables. These estimates constitute the values of safety factor calculated for all possible combinations of $x_+$ and $x_-$ for all input variables. Commonly, $x_+$ and $x_-$ are taken one standard deviation above and one standard deviation below the mean (U.S. Army, 1992). Before summing, the individual terms are multiplied by corresponding probability concentrations which are functions of correlation coefficients between variables. As the number of input variables increases, the number of terms to be evaluated increase by a power law and the analysis gets more cumbersome. The mathematical details of the technique can be found in Rosenblueth (1975, 1981) and Harr (1987). While the method is shown to be reasonably accurate for a wide range of practical problems, it can be seriously in error in some cases.
Uncertainty and probability in slope assessments

Monte Carlo Simulations – Key points

• Aims at simulating a large number of scenarios of FoS for the possible combination of parameter values, according to each parameter’s probability distribution.

• Each scenario can be a deterministic equation for the FoS or a “run” for a limit equilibrium model or SR model (although this last one with an increased amount of computation effort).

• The method selects the parameter values randomly (through random number generation). Correlation between parameters can also be added in some computational packages.

• The result is the aggregated results of FoS for all scenarios, which is treated as an observation of test results and a probability distribution of FoS.

After US DOT
Monte Carlo Simulations – Key points

• An advantage is that it provides the full shape of the PDF, eliminating the assumptions of the shape of this PDF.
• The more complex the model, the more computational effort required
• The more parameters treated stochastically (with a PDF) the more scenarios (or iterations) are needed.

DON’T FORGET THE SPATIAL CORRELATIONS! SIGNIFICANT EFFECT ON POF!!!
Slope Stability Analysis in a Random Soil

Two simulations of a random slope stability analysis by RFEM. Both slopes have the same mean and standard deviation.
What is the relationship between FoS and PoF?

The “margin” for safety ($M$) can be quantified as:

$$M = R - Q$$

Where $M \leq 0$ represents failure. If $R$ and $Q$ are defined by their probability distributions:

$$\mu_M = \mu_R - \mu_Q; \quad \sigma_M = \sqrt{\sigma_R^2 + \sigma_Q^2 - 2\rho_{RQ}\sigma_R\sigma_Q}$$

$$\rho_{RQ} = \frac{E[(R-\mu_R)(Q-\mu_Q)]}{\sigma_R\sigma_Q} \quad (Correlation\ coefficient\ between\ R\ and\ Q)$$

$$\beta = \frac{\mu_M}{\sigma_M} \quad (Reliability\ Index)$$
Reliability, FoS and PoF

The Factor of Safety (FoS) is the common metric in slope stability analyses

$$FoS = \frac{R}{Q}$$

and the Reliability Index:

$$\beta = \frac{E[FoS] - 1}{\sigma_{FoS}}$$

When using FoS, previous equations are not applicable, unless R and Q are assumed LogNormally distributed. Using the logarithms of R and Q and assuming them Normally distributed will validate the previous equations for FoS

Baecher and Christian, 2003
This provides a means to calculate the relationship between FoS and PoF

\[ 1 + \beta \times \sigma_{FoS} = E[FoS] \]

This implies that the relationship between FoS and PoF depends on the “spread” of the FoS distribution.

<table>
<thead>
<tr>
<th>Reliability index</th>
<th>Normal distribution</th>
<th>Triangular distribution</th>
<th>LogNormal distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( 5.000 \times 10^{-1} )</td>
<td>( 5.000 \times 10^{-1} )</td>
<td>( 5.100 \times 10^{-1} )</td>
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<td>( 2.867 \times 10^{-7} )</td>
<td>0.0</td>
<td>( 4.955 \times 10^{-9} )</td>
</tr>
</tbody>
</table>

Baecher and Christian, 2003

\[ COV(\Omega) = \frac{s}{\mu} = \frac{\sigma_{FoS}}{E[FoS]} - 1 \]
Reliability, FoS and PoF

This plot implies that reduction of uncertainty through enhanced field investigation, increased monitoring and back analyses, and improved modelling techniques; allows a *safer* and *more economic* design.

El-Ramly, 2001
Role of monitoring and performance

- Monitoring is a very mature practice in Open Pit mining.
- This provides a continuous measure of slope performance through the life of the pit.
- Measured displacements and slope deformations can be used to set criteria for slope performance, and in turn, slope geometry for future pushbacks.
- This can be as measures of slope strain ($\delta/H_{\text{slope}} \times 100$) and qualitative observations of slope deformation (elastic rebound vs. plastic deformations; constant deformation velocity vs. sustained acceleration).

These are case specific and based on site experience and will not be discussed further.
Role of monitoring and performance

Monitoring slope performance in slope management - OM

Observational Method (OM):

1. Design under the most likely conditions. Acceptable limits of behaviour are established;
2. Worse but plausible ground conditions need to be considered to assess other potential underperformance mechanisms and those parameters that will indicate the occurrence of such scenario;
3. Design, construction, or operation modifications / enhancements are devised for the event that ground response deviates from the ranges of possible behaviour;
4. Monitoring is devised which will reveal whether the actual behaviour lies within the acceptable limits;
5. The response time of the instruments and the procedures for analysing the results shall be sufficiently rapid in relation to the possible evolution of the instability;
6. During construction and operations, the monitoring shall be carried out as planned;
7. The result of monitoring shall be assessed at appropriate stages and the planned contingency actions shall be put into operation if the limits of behaviour are exceeded;
8. Monitoring equipment shall either be replaced or extended if it fails to supply reliable data of appropriate type or in sufficient quality.
Role of monitoring and performance

What is the impact of monitoring slope performance in PoF of subsequent pushbacks?

This allows a safer, more efficient design

El-Ramly, 2001
In summary:

- Uncertainty governs most (or all) steps for rock slope design, excavation, management and closure.
- Be mindful of all potential sources of uncertainty, learn how to indent them, and know the tools available to quantify and reduce them.
- More important, know the limitations of these tools!
- Understand the design criteria. Is it based on uncertainty? Are consequences too high? Is it about deformations?
- Understand what your instrumentation is telling you. Measure and assess performance in the basis of your understanding. Do not just measure and report.
- What are your goals?
REFERENCES

• Griffiths, 2018, Slope Stability Analysis by Finite Elements. Presentation for Plan 111, Chang’an University, Xi’an, China
• Martin, D. & Stacey, P. eds., 2017. Guidelines for Open Pit Slope Design in Weak Rocks, CSIRO
• Read, J. & Stacey, P. eds., 2009. Guidelines for Open Pit Slope Design, CSIRO.
Floor open for discussion